

CUBIC THREEFOLDS, FANO SURFACES AND THE MONODROMY OF THE GAUSS MAP

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ABSTRACT. The Tannakian formalism allows to attach to any subvariety of an abelian variety an algebraic group in a natural way. The arising groups are closely related to moduli questions such as the Schottky problem, but in general they are still poorly understood. In this note we show that for the theta divisor on the intermediate Jacobian of a cubic threefold, the Tannaka group is exceptional of type E_6 . This is the first known exceptional case, and it suggests a surprising connection with the monodromy of the Gauss map.

INTRODUCTION

Let A be a complex abelian variety. To any closed subvariety $Z \hookrightarrow A$ one may attach a semisimple complex algebraic group G_Z in a natural way, using the Tannakian formalism of [16] [20] as we explain in section 1. These Tannaka groups are closely related to classical topics such as the Schottky problem [18], intersections of theta divisors [17], the Torelli theorem [26] and Brill-Noether theory [23]. In all previously known examples they are classical groups, and they are simply connected unless Z is a sum of two positive-dimensional subvarieties of A . This raises several questions: (1) Are there examples where G_Z is an exceptional group? (2) If G_Z is not simply connected, then is its simply connected cover always realized as the Tannaka group of some other subvariety? (3) What sort of geometric information is encoded in these groups? In this note we discuss a family of principally polarized abelian varieties of dimension five which gives a positive answer to the first question, the intermediate Jacobians of smooth cubic threefolds [8]:

Theorem 1. *Let $\Theta \subset A$ be the theta divisor on the intermediate Jacobian A of a smooth cubic threefold. Then*

$$G_\Theta \cong E_6(\mathbb{C})/Z,$$

where Z denotes the center of the simply connected complex algebraic group $E_6(\mathbb{C})$.

A more precise statement will be given in theorem 2. In particular, the universal covering group $E_6(\mathbb{C})$ is realized as the Tannaka group for the Fano surface of lines on the threefold so that the second of the above questions also has a positive answer in this example. Concerning the third question, recall that the Weyl group $W(E_6)$ is the symmetry group of the 27 lines on a cubic surface. The proof of theorem 2 will not use this explicitly, indeed it needs surprisingly little geometry and rests on representation theory, see lemma 4. But the result suggests that for any smooth subvariety $Z \hookrightarrow A$ the Weyl group $W(G_Z)$ should contain the monodromy group of

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a certain Gauss map as a subgroup of small index, see section 4; this would explain the relevance of the above Tannaka groups for classical moduli problems.

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1. THE TANNAKIAN FRAMEWORK

Let $D(A) = D_c^b(A, \mathbb{C})$ be the bounded derived category of constructible sheaves of complex vector spaces on a complex abelian variety A , and $P(A) \subset D(A)$ its full abelian subcategory of perverse sheaves [4] [15]. The group law $a : A \times A \rightarrow A$ endows these with a rich structure: The convolution product

$$* : P(A) \times P(A) \longrightarrow D(A), \quad \delta_1 * \delta_2 = Ra_*(\delta_1 \boxtimes \delta_2)$$

leads to a Tannakian description for the category of perverse sheaves in terms of group representations [16] [20]. There is a slight technicality here since in general the convolution of perverse sheaves is no longer perverse. To get around this, let us say a perverse sheaf $\delta \in P(A)$ with hypercohomology $H^\bullet(A, \delta)$ is *negligible* if the Euler characteristic

$$\chi(\delta) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} H^i(A, \delta)$$

vanishes. The negligible perverse sheaves have been classified in [24] [21, cor. 5.2] and may be described explicitly in terms of perverse sheaves on proper abelian quotient varieties of A . If $N(A) \subset D(A)$ denotes the full subcategory of all sheaf complexes whose perverse cohomology sheaves are negligible, it has been shown in [20] that the triangulated quotient category $D(A)/N(A)$ inherits a well-defined convolution product $*$ and that this latter product preserves the full abelian subcategory

$$\overline{P}(A) = P(A)/(N(A) \cap P(A)) \subset D(A)/N(A).$$

Furthermore $\overline{P}(A)$ is a direct limit of Tannakian categories: The subquotients of the convolution powers of any given perverse sheaf $\delta \in \overline{P}(A)$ generate a full abelian subcategory $\langle \delta \rangle \subset \overline{P}(A)$ which is stable under the convolution product and admits an equivalence

$$\omega : \langle \delta \rangle \xrightarrow{\sim} \text{Rep}(G)$$

with the Tannakian category of finite-dimensional complex linear representations of a complex algebraic group $G = G(\delta)$. In particular, $\omega(\delta_1 * \delta_2) \cong \omega(\delta_1) \otimes \omega(\delta_2)$ for all $\delta_1, \delta_2 \in \langle \delta \rangle$. It follows from the constructions that the dimension of the arising representations is the Euler characteristic: $\dim_{\mathbb{C}} \omega(\delta) = \chi(\delta)$.

In principle this reduces the study of perverse sheaves on A to the representation theory of algebraic groups, though it may be hard to determine the groups $G(\delta)$ explicitly. The most interesting case arises when $\delta = \delta_Z$ is taken to be the perverse intersection cohomology sheaf supported on a closed subvariety $Z \hookrightarrow A$. Often Z is only determined up to a translation by some point of the abelian variety, but the group $G(\delta_Z)$ depends on the chosen translate. So instead of this group we consider the semisimple group $G_Z = [G(\delta_Z)^\circ, G(\delta_Z)^\circ]$ which is the derived group of the connected component. This derived group remains unaffected if Z is replaced by a translate. Let $\omega_Z = \omega(\delta_Z)|_{G_Z} \in \text{Rep}(G_Z)$ denote its defining representation.

2. INTERMEDIATE JACOBIANS

Let $V \subset \mathbb{P}^4$ be a smooth cubic threefold. Clemens and Griffiths have shown [8] that the Fano surface S of lines on V is smooth with $\chi(\delta_S) = 27$, embeds in the intermediate Jacobian

$$JV = \text{Hom}(H^{2,1}(V), \mathbb{C})/H_3(V, \mathbb{Z}),$$

and that the latter is a principally polarized abelian variety of dimension $g = 5$ which admits as a theta divisor the image $\Theta = S - S \subset JV$ of the difference morphism $d : S \times S \rightarrow JV, (s, t) \mapsto s - t$. This theta divisor has an isolated singularity at the origin, and there its projective tangent cone is isomorphic to V by [3]. One then computes $\chi(\delta_\Theta) = 78$, see section 3.

The numbers 27 and 78 also arise in representation theory as the dimensions of the smallest irreducible representations of the simply connected complex algebraic group $E_6(\mathbb{C})$. Up to duality they are attained precisely for the 27-dimensional first fundamental representation ω_1 and the 78-dimensional adjoint representation Ad of this group on its Lie algebra. Note that the adjoint representation factors over the quotient $E_6(\mathbb{C})/Z$ by the center $Z = Z(E_6(\mathbb{C}))$.

Theorem 2. *Let A be the intermediate Jacobian of a smooth cubic threefold, and consider the corresponding Fano surface $S \subset A$ and the theta divisor $\Theta = S - S \subset A$ as above. Then*

$$(G_S, \omega_S) \cong (E_6(\mathbb{C}), \omega_1) \quad \text{and} \quad (G_\Theta, \omega_\Theta) \cong (E_6(\mathbb{C})/Z, Ad).$$

Proof. We have already remarked that $\Theta \subset A$ is the image of the difference morphism $d : S \times S \rightarrow A$. The latter is generically finite, so by adjunction it follows from the decomposition theorem that δ_Θ occurs as a direct summand of the convolution $Rd_*(\delta_S \boxtimes \delta_S) = \delta_S * \delta_{-S}$. On the Tannakian side this gives an embedding

$$\omega_\Theta \hookrightarrow \omega_S \otimes \omega_S^\vee \quad \text{for the dual} \quad \omega_S^\vee = \text{Hom}(\omega_S, \mathbb{C}),$$

so that G_Θ becomes identified with the image of the Tannaka group G_S under the tensor product representation $\omega_S \otimes \omega_S^\vee$. The rest of the proof is an application of representation theory which may be found in lemma 4 below. \square

Remark 3. A result of Collino [9] says that in the moduli space \mathcal{A}_5 of principally polarized abelian fivefolds, the closure of the locus of intermediate Jacobians of smooth cubic threefolds contains the locus of Jacobians of hyperelliptic curves. See also [7]. For a degeneration of a general intermediate Jacobian into a hyperelliptic one, the Fano surface S degenerates to the Brill-Noether subvariety W_2 . The latter is by definition the image of the symmetric square of the curve in its Jacobian variety, and it has the Tannaka group $G_{W_2} \cong Sp_8(\mathbb{C})/\pm 1$ by [19, th. 6.1] [23]. By semicontinuity [18, sect. 4] this group must be a subquotient of the Tannaka group for the Fano surface, which gives a consistency check for our results and could be used for an alternative proof of theorem 2:

$$\begin{array}{ccc} Sp_8(\mathbb{C}) & \hookrightarrow & E_6(\mathbb{C}) = G_S \\ \downarrow & & \\ G_{W_2} = Sp_8(\mathbb{C})/\pm 1 & & \end{array}$$

Lemma 4. *If G is a connected semisimple complex algebraic group which admits a faithful irreducible representation $V \in \text{Rep}(G)$ of dimension 27 such that $V \otimes V^\vee$ contains an irreducible direct summand $W \in \text{Rep}(G)$ of dimension 78, then up to duality*

$$G \cong E_6(\mathbb{C}), \quad V \cong \omega_1 \quad \text{and} \quad W \cong \text{Ad}.$$

Proof. To any connected semisimple algebraic group G one may associate its universal covering

$$\tilde{G} = G_1 \times \cdots \times G_n \twoheadrightarrow G,$$

where each G_i is simply connected and simple modulo its center. It then follows that

$$V|_{\tilde{G}} \cong V_1 \boxtimes \cdots \boxtimes V_n$$

is an exterior product of irreducible representations $V_i \in \text{Rep}(G_i)$ with finite kernel and dimension $d_i = \dim(V_i) > 1$. Having $d_1 \cdots d_n = \dim(V) = 27$ forces that $n \leq 3$ and $d_i \in \{3, 9, 27\}$. Now for complex simple Lie algebras of any Dynkin type, the highest weights of all irreducible representations of a given (small) dimension are easily determined via the result of [1], see table 1 where we denote by $\omega_1, \omega_2, \dots$ the fundamental weights. In dimensions $d_i \in \{3, 9, 27\}$ we are left with the following representations and their duals:

d_i	3		9			27								
G_i	A_1	A_2	A_1	A_8	B_4	A_1	A_2	A_{26}	B_3	B_{13}	C_4	E_6	G_2	
V_i	$2\omega_1$	ω_1	$8\omega_1$	ω_1	ω_1	$26\omega_1$	$2\omega_1 + 2\omega_2$	ω_1	$2\omega_1$	ω_1	ω_2	ω_1	$2\omega_1$	

In our case each $V_i \otimes V_i^\vee$ must by assumption have an irreducible direct summand W_i such that

$$W|_{\tilde{G}} \cong W_1 \boxtimes \cdots \boxtimes W_n.$$

Since $\dim(W) = 78$, a direct computation shows that the only possibility is $n = 1$ and that the universal covering group $\tilde{G} = G_1$ is isomorphic to $E_6(\mathbb{C})$. \square

3. THE MAGIC NUMBER 78

It remains to check that for the theta divisor on the intermediate Jacobian of a smooth cubic threefold we have $\chi(\delta_\Theta) = 78$. This is a simple computation, but we include it since the outcome is crucial for our proof of theorem 2. We begin with a blowup formula. Let A be a complex abelian variety, and consider an effective divisor $D \subset A$ whose singular locus $D^{sg} = \{p_1, \dots, p_n\}$ consists of finitely many isolated points p_i of multiplicity m_i . Blowing up these finitely many points in A we obtain a Cartesian diagram

$$\begin{array}{ccccc} E & \hookrightarrow & \tilde{D} \cup E & \hookrightarrow & \tilde{A} \\ \downarrow & & \downarrow & & \downarrow \pi \\ D^{sg} & \hookrightarrow & D & \hookrightarrow & A \end{array}$$

where E denotes the exceptional divisor of the blowup and where \tilde{D} is the strict transform of the divisor D . The restriction $\pi : \tilde{D} \rightarrow D$ is the blowup of our original divisor in the singular locus and so the scheme-theoretic intersection $E \cap \tilde{D} \subset \tilde{D}$ is a Cartier divisor. In particular, if the latter is smooth, then \tilde{D} is smooth.

Proposition 5. *If \tilde{D} is smooth, then*

$$(-1)^{g-1} \chi(\tilde{D}) = \deg [D]^g - \sum_{i=1}^n m_i \left[\frac{(1+t)(1-t)^g}{1-m_i t} \right]_{t^{g-1}}$$

where $\deg : H^{2g}(A, \mathbb{Z}) \rightarrow \mathbb{Z}$ denotes the degree map and where the brackets on the right hand side indicate the coefficient of t^{g-1} in the power series.

Proof. Let $E_i = \pi^{-1}(p_i) \subset \tilde{A}$ denote the exceptional fibres of the blowup, and consider the fundamental cohomology classes $\eta_i = [E_i]$, $\theta = [\tilde{D}]$ in $H^2(\tilde{A}, \mathbb{Q})$. Since the tangent bundle to A is trivial, the formula for the total Chern class of a blowup in [13, ex. 15.4.2] says that $c(\tilde{A}) = (1+\eta)(1-\eta)^g$ where $\eta = \sum_i \eta_i$. So the conormal sequence together with the projection formula for the embedding $\iota : \tilde{D} \hookrightarrow \tilde{A}$ gives the formula $\iota_*(c(\tilde{D})) = (1+\eta)(1-\eta)^g \cdot \theta \cdot (1+\theta)^{-1}$ for the total Chern class of the proper transform. The Gauss-Bonnet theorem says that $\chi(\tilde{D})$ is the degree g part of $\iota_*(c(\tilde{D}))$, so our claim follows by writing

$$\theta = \pi^*[D] - \sum_{i=1}^n m_i \cdot \eta_i$$

and using that $\eta_i \cdot \eta_k = 0$ for $i \neq k$ and $\eta_i \cdot \pi^*[D] = 0$ whereas $\deg \eta_i^g = (-1)^{g-1}$ for all i . This last intersection number is obtained from the conormal sequence for the smooth divisor $E_i \cong \mathbb{P}^{g-1}$ with $\deg c_{g-1}(E_i) = \chi(E_i) = g$. \square

Corollary 6. *The theta divisor $\Theta \subset A$ on the intermediate Jacobian $A = JV$ of a smooth cubic threefold V has*

$$\chi(\delta_\Theta) = 78.$$

Proof. Let $\pi : \tilde{\Theta} \rightarrow \Theta$ denote the blowup of the theta divisor in the origin. The fibre $\pi^{-1}(0)$ is isomorphic to our threefold V by [3, th. 1], so base change implies that the stalk cohomology of $R\pi_*(\delta_{\tilde{\Theta}})$ in the origin is $\mathcal{H}^i(R\pi_*(\delta_{\tilde{\Theta}}))_0 \cong H^{i+4}(V, \mathbb{C})$ for all $i \in \mathbb{Z}$. On the other hand

$$R\pi_*(\delta_{\tilde{\Theta}}) \cong \delta_\Theta \oplus \varepsilon$$

by the decomposition theorem, where ε is a skyscraper complex which is supported in the origin and is stable under the Lefschetz operator and its inverse. Since the stalk cohomology of δ_Θ vanishes in all degrees $i \geq 0$, a comparison with $H^{i+4}(V, \mathbb{C})$ shows $\varepsilon \cong \delta_0[2] \oplus \delta_0 \oplus \delta_0[-2]$ and hence $\chi(\delta_{\tilde{\Theta}}) = \chi(\delta_\Theta) + 3$. Therefore our claim follows from the above proposition which in the special case $n = 1$, $m_1 = 3$ shows that $\chi(\delta_{\tilde{\Theta}}) = 5! - 39 = 81 = 78 + 3$. \square

Remark 7. (a) The Euler characteristic of a perverse sheaf has a simple meaning in terms of Gauss maps. To explain this, let A be any complex abelian variety, and put $\Omega = H^0(A, \Omega_A^1)$. For any closed subvariety $Z \hookrightarrow A$, the closure of the conormal bundle to the smooth locus of Z is an irreducible subvariety $\Lambda_Z \hookrightarrow \mathcal{T}_A^* = A \times \Omega$ of the cotangent bundle to the abelian variety. We define the corresponding Gauss map to be the composite

$$\gamma_Z : \Lambda_Z \hookrightarrow \mathcal{T}_A^* = A \times \Omega \rightarrow \Omega,$$

and we denote the degree of this generically finite map by $\deg(\gamma_Z) \in \mathbb{N} \cup \{0\}$. Now to any perverse sheaf $\delta \in P(A)$ we may attach a regular holonomic \mathcal{D}_A -module

by the Riemann-Hilbert correspondence [15], and its characteristic cycle is a finite formal sum

$$\mathrm{CC}(\delta) = \sum_{Z \hookrightarrow A} m_Z(\delta) \cdot \Lambda_Z$$

with coefficients $m_Z(\delta) \in \mathbb{N} \cup \{0\}$. Franecki and Kapranov [12] have shown that in this situation

$$\chi(\delta) = \sum_{Z \hookrightarrow A} m_Z(\delta) \cdot \deg(\gamma_Z).$$

If $\delta = \delta_W$ is taken to be the perverse intersection cohomology sheaf supported on a closed subvariety $W \hookrightarrow A$, then $m_W(\delta) = 1$, but depending on how bad the singularities of the subvariety are, we may also have $m_Z(\delta) > 0$ for some $Z \hookrightarrow W^{sg}$ inside the singular locus.

(b) The theta divisor $\Theta \subset A$ on the intermediate Jacobian A of a smooth cubic threefold has an isolated singularity at the origin, so $\mathrm{CC}(\delta_\Theta) = \Lambda_\Theta + m \cdot \Lambda_{\{0\}}$ for some $m \geq 0$. To compute the multiplicity m , recall that by [8, sect. 13] we have a commutative diagram of rational maps

$$\begin{array}{ccc} S \times S & \xrightarrow{\Psi} & \Theta = \mathbb{P}\Lambda_\Theta \\ & \searrow \Phi & \downarrow \mathbb{P}\gamma_\Theta \\ & & \mathbb{P}^4 = \mathbb{P}\Omega \end{array}$$

where Ψ has generic degree six and where the generic fibre of Φ can be identified with the set of $27 \cdot 16 = 432$ pairs of skew lines on the smooth cubic surface arising as the generic hyperplane section of the threefold. Note that there is a counting error in loc. cit. in the paragraph after (13.7); there all $27 \cdot 26$ pairs of lines are considered, but the non-skew pairs are mapped by Φ to a proper closed subset of \mathbb{P}^4 and do not contribute to the generic fibre. With the corrected number we get $\deg(\gamma_\Theta) = 432/6 = 72$, so corollary 6 and the formula of Franecki and Kapranov imply

$$\mathrm{CC}(\delta_\Theta) = \Lambda_\Theta + m \cdot \Lambda_{\{0\}} \quad \text{with} \quad m = 78 - 72 = 6.$$

The nontrivial contribution of the singular locus may be surprising since irreducible theta divisors have only mild singularities [11].

4. THE MONODROMY OF THE GAUSS MAP

Together with the previously known examples, theorem 2 suggests the following geometric interpretation for the Tannaka groups G_Z from section 1.

Conjecture 8. *For any smooth closed subvariety Z of an abelian variety A , the Weyl group $W_Z = W(G_Z)$ contains as a subgroup of small index the monodromy group M_Z of the Gauss map $\gamma_Z : \Lambda_Z \rightarrow \Omega$.*

Here the monodromy group is defined as follows. By [25] [21, cor. 5.2] the Gauss map γ_Z is dominant unless Z is stable under translations by all points of a non-zero abelian subvariety, in which case $G(\delta_Z) = \{1\}$. Discarding this trivial case, we may assume that the Gauss map restricts over an open dense subset of Ω to a finite étale cover of degree $\deg(\gamma_Z) = \chi(\delta_Z) > 0$. We define its monodromy group M_Z to be the Galois group of the Galois hull of this cover, which is isomorphic to the image of the monodromy representation on a general fibre of γ_Z .

Theorem 9. *We have the following monodromy and Weyl groups:*

- (1) *If A is the Jacobian variety of a smooth projective curve $C \hookrightarrow A$ of genus g , then*

$$M_C \cong W_C \cong \begin{cases} (\pm 1)^{g-1} \rtimes \mathfrak{S}_{g-1} & \text{if } C \text{ is hyperelliptic,} \\ \mathfrak{S}_{2g-2} & \text{otherwise.} \end{cases}$$

- (2) *If A is the intermediate Jacobian of a smooth cubic threefold with Fano surface $S \hookrightarrow A$, then*

$$M_S \cong W_S \cong W(E_6).$$

- (3) *If A is a general principally polarized abelian variety of dimension $g > 2$ with theta divisor $\Theta \hookrightarrow A$, then*

$$M_\Theta \cong (\pm 1)_0^r \rtimes \mathfrak{S}_r \quad \text{and} \quad W_\Theta \cong \begin{cases} (\pm 1)^r \rtimes \mathfrak{S}_r & \text{if } 2 \mid g, \\ (\pm 1)_0^r \rtimes \mathfrak{S}_r & \text{if } 2 \nmid g, \end{cases}$$

where $r = g!/2$ and where $(\pm 1)_0^r = \{(\epsilon_1, \dots, \epsilon_r) \in (\pm 1)^r \mid \epsilon_1 \cdots \epsilon_r = 1\}$.

Proof. For the Weyl groups this follows from theorem 2 and [19, th. 6.1] [18], so it only remains to discuss the monodromy groups. For the Jacobian varieties in (1) we have $\Omega = H^0(C, \Omega_C^1)$ and

$$\Lambda_C = \{(p, \omega) \in C \times \Omega \mid \omega(p) = 0\} \hookrightarrow \mathcal{T}_A^* = A \times \Omega.$$

Let $\iota : C \rightarrow \mathbb{P}\Omega^*$ be the canonical map and $\overline{C} = \iota(C)$ its image. With the usual identification of points in a projective space and hyperplanes in the dual space, this gives the following factorization for the projectivized Gauss map:

$$\mathbb{P}\Lambda_C = \{(p, H) \in C \times \mathbb{P}\Omega \mid \iota(p) \in H\} \xrightarrow{\alpha} \{(\overline{p}, H) \in \overline{C} \times \mathbb{P}\Omega \mid \overline{p} \in H\} \xrightarrow{\beta} \mathbb{P}\Omega$$

By the uniform position principle [2, lemma on p. 111] the monodromy group of β is the symmetric group \mathfrak{S}_d of degree $d = \deg(\overline{C})$. In the non-hyperelliptic case we have $d = 2g - 2$ and α is an isomorphism, so we are done. It remains to discuss the hyperelliptic case. In that case $d = g - 1$ and α is the quotient by the hyperelliptic involution. So the general fibre of the Gauss map consists of $g - 1$ pairs of points that are interchanged under the hyperelliptic involution. The monodromy M_C is then a subgroup of the semidirect product $(\pm 1)^{g-1} \rtimes \mathfrak{S}_{g-1}$ which surjects onto the quotient \mathfrak{S}_{g-1} via the induced permutation action on a general fibre of β . It remains to show that for any of the $g - 1$ pairs of points in a general fibre of the Gauss map, the group M_C contains a permutation which interchanges the two points of this pair but fixes all the other points in the fibre. But this follows from the observation that through any branch point $\overline{p} \in \overline{C}$ of the double cover $C \rightarrow \overline{C} \cong \mathbb{P}^1$ there exists a hyperplane $H \in \mathbb{P}\Omega$ which does not meet any other branch point of this cover and over which furthermore the cover β is unramified, meaning that this hyperplane is nowhere tangent to the curve \overline{C} . Hence part (1) follows.

For the intermediate Jacobian A of a smooth cubic threefold V in (2), recall that the Fano surface $S \subset A$ parametrizes the lines on the threefold. In fact there exists by [3, prop. 6] an embedding $V \hookrightarrow \mathbb{P}\Omega^* = \mathbb{P}T_0(A)$ such that for any $p \in S(\mathbb{C})$ the

corresponding line $L_p \subset V$ is identified with the projective tangent space to the Fano surface at that point:

$$\begin{array}{ccccc} L_p & \hookrightarrow & V & \hookrightarrow & \mathbb{P}\Omega^* \\ \parallel & & & & \parallel \\ \mathbb{P}T_p(S) & \hookrightarrow & & \hookrightarrow & \mathbb{P}T_p(A) \end{array}$$

Identifying points $H \in \mathbb{P}\Omega$ with hyperplanes $H \subset \mathbb{P}\Omega^*$ in the dual projective space, we obtain that

$$\mathbb{P}\Lambda_S = \{(p, H) \in S \times \mathbb{P}\Omega \mid L_p \subset H \cap V\}.$$

So the fibre of the Gauss map γ_S over a general point $H \in \mathbb{P}\Omega$ is identified with the 27 lines on the smooth cubic surface $H \cap V$, and the monodromy group M_S is the group of permutations of these 27 lines which is induced by variations of the hyperplane $H \in \mathbb{P}\Omega$. By [22, VI.20] this group is $W(E_6)$, so (2) follows.

For part (3) we take a translate of the theta divisor $\Theta \subset A$ which is stable under the involution $\iota = -id : A \rightarrow A$. The projectivized Gauss map factors over the quotient by this involution:

$$\mathbb{P}\Lambda_\Theta = \Theta \xrightarrow{\alpha} \Theta/\langle\iota\rangle \xrightarrow{\beta} \mathbb{P}\Omega.$$

From [10, sect. 1] we know that over a general point $H \in \mathbb{P}\Omega$ in the branch locus of the generically finite map β , the fibre $\beta^{-1}(H)$ will consist of precisely $r - 1$ distinct points where $r = \deg(\beta) = g!/2$. Furthermore α is étale over the complement of the finitely many singular points of $\Theta/\langle\iota\rangle$. Restricting $\beta \circ \alpha$ to a general line $\mathbb{P}^1 \hookrightarrow \mathbb{P}\Omega$ we get by [14, th. 1.1] a branched cover

$$X = \Theta \times_{\mathbb{P}\Omega} \mathbb{P}^1 \xrightarrow{\alpha'} X/\langle\iota\rangle \xrightarrow{\beta'} \mathbb{P}^1$$

of irreducible curves such that the monodromy group of the cover $\alpha' \circ \beta'$ coincides with the one of $\alpha \circ \beta$ (the inclusion of a general line induces an epimorphism on the fundamental groups of the complements of the branch loci). For $g > 2$ we may assume that the double cover α' is étale and that β' is only simply ramified, and our claim follows from [5, th. 1] after passing to the normalizations of the respective curves. Note that the situation for $g = 2$ is different but covered by (1). \square

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d	A_2	A_3	A_4	A_5	A_6	A_7	B_3	B_4	C_2	C_3	C_4	D_4	D_5	E_6	F_4	G_2
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
5	—	—	—	—	—	—	—	—	ω_2	—	—	—	—	—	—	—
6	$2\omega_1$	ω_2	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	ω_1
8	$\omega_1 + \omega_2$	—	—	—	—	—	ω_3	—	—	—	—	ω_3, ω_4	—	—	—	—
9	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
10	$3\omega_1$	$2\omega_1$	ω_2	—	—	—	—	—	$2\omega_1$	—	—	—	—	—	—	—
11	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
12	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
13	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
14	—	—	—	—	—	—	—	—	$2\omega_2$	ω_2, ω_3	—	—	—	—	—	ω_2
15	$4\omega_1, 2\omega_1 + \omega_2$	$\omega_1 + \omega_3$	$2\omega_1$	ω_2	—	—	—	—	—	—	—	—	—	—	—	—
16	—	—	—	—	—	—	—	ω_4	$\omega_1 + \omega_2$	—	—	—	ω_4	—	—	—
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
20	—	$3\omega_1, 2\omega_2,$ $\omega_1 + \omega_2$	—	ω_3	—	—	—	—	$3\omega_1$	—	—	—	—	—	—	—
21	$5\omega_1$	—	—	$2\omega_1$	ω_2	—	ω_2	—	—	$2\omega_1$	—	—	—	—	—	—
22	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
23	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
24	$3\omega_1 + \omega_2$	—	$\omega_1 + \omega_4$	—	—	—	—	—	—	—	—	—	—	—	—	—
25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
26	—	—	—	—	—	—	—	—	—	—	—	—	—	—	ω_4	—
27	$2\omega_1 + 2\omega_2$	—	—	—	—	—	$2\omega_1$	—	—	—	ω_2	—	—	ω_1	—	$2\omega_1$
28	$6\omega_1$	—	—	—	$2\omega_1$	ω_2	—	—	—	—	—	ω_2	—	—	—	—
29	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
30	—	—	—	—	—	—	—	—	$3\omega_2$	—	—	—	—	—	—	—

TABLE 1. Up to duality, all irreducible representations of dimension $d \leq 30$ for the complex simple Lie algebras of type $\neq A_1$, with the exception of the defining representations of the classical Lie algebras of type A_n, B_n, C_n, D_n with $d = n+1, 2n+1, 2n, 2n$. We denote the representations by their highest weights, using the fundamental weights $\omega_1, \omega_2, \dots$ labelled as in [6].

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